ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF SEMILINEAR NONAUTONOMOUS EQUATIONS

NGUYEN VAN MINH, GASTON M. N'GUÉRÉKATA, AND CIPRIAN PREDA

Abstract. We consider nonautonomous semilinear evolution equations of the form

$$\frac{dx}{dt} = A(t)x + f(t,x)\ . \tag{1}$$

Here $A(t)$ is a (possibly unbounded) linear operator acting on a real or complex Banach space $\mathbb{X}$ and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a (possibly nonlinear) continuous function. We assume that the linear equation (3) is well-posed (i.e. there exists a continuous linear evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ such that for every $s \in \mathbb{R}_+$ and $x \in D(A(s))$, the function $x(t) = U(t,s)x$ is the uniquely determined solution of equation (3) satisfying $x(s) = x$). Then we can consider the **mild solution** of the semilinear equation (4) (defined on some interval $[s, s+\delta), \delta > 0$) as being the solution of the integral equation

$$x(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau, x(\tau))d\tau \quad , \quad t \geq s\ , \tag{2}$$

Furthermore, if we assume also that the nonlinear function $f(t,x)$ is jointly continuous with respect to t and x and Lipschitz continuous with respect to x (uniformly in $t \in \mathbb{R}_+$, and $f(t,0) = 0$ for all $t \in \mathbb{R}_+$) we can generate a (nonlinear) evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$, in the sense that the map $t \mapsto X(t,s)x : [s,\infty) \to \mathbb{X}$ is the unique solution of equation (6), for every $x \in \mathbb{X}$ and $s \in \mathbb{R}_+$.

Considering the *Green's operator* $(\mathbb{G}f)(t) = \int_0^t X(t,s)f(s)ds$ we prove that if the following conditions hold

- the map $\mathbb{G}f$ lies in $L^q(\mathbb{R}_+, \mathbb{X})$ for all $f \in L^p(\mathbb{R}_+, \mathbb{X})$, and
- $\mathbb{G} : L^p(\mathbb{R}_+, \mathbb{X}) \to L^q(\mathbb{R}_+, \mathbb{X})$ is Lipschitz continuous, i.e. there exists $K > 0$ such that

$$\|\mathbb{G}f - \mathbb{G}g\|_q \leq K\|f - g\|_p\ ,\ \text{for all } f, g \in L^p(\mathbb{R}_+, \mathbb{X})\ ,$$

then the above mild solution will have an exponential decay.

1. Introduction

In the last few decades, we can note an increasing research interest in the asymptotic behavior of the solutions of the linear differential equations

$$\frac{dx}{dt} = A(t)x\ ,\ t \in [0,\infty)\ ,\ x \in \mathbb{X}\ , \tag{3}$$

where $A(t)$ is in general an unbounded linear operator on a Banach space $\mathbb{X}$, for every fixed t.

In the case that $A(t)$ is a matrix continuous function, O. Perron [37] first observed a connection between the asymptotic behavior of the solutions of the above equation and the properties of the differential operator $\frac{d}{dt} - A(t)$ as an operator on the space $C_b(\mathbb{R}_+, \mathbb{R}^n)$ of R^n-valued, bounded and continuous functions on the half-line $\mathbb{R}_+$.

This result became a milestone for many works on the qualitative theory of solutions of differential equations. We refer the reader to the monograph by Massera and Schäffer [25], and Daleckij and Krein [11] for a characterization of the exponential dichotomy of the solutions of the above equation in terms of surjectiveness of the differential operator $\frac{d}{dt} - A(t)$ in the case of bounded $A(t)$ and by Levitan and Zhikov [23] for an extension to the infinite-dimensional setting for equations defined on the whole line.

Recently, there is a lot of work done in the study of the asymptotic behavior of the solutions of differential equations in Banach spaces, in particular in the unbounded case (see e.g., [13, 32, 38, 39, 47]).

Also, there is an increasing interest "applications-wise" for the Cauchy problems associated with the above evolution equations since many physical situations can be interpreted as Cauchy problems by choosing the "right" state space. Among the physical equations which are eligible for such an approach

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we can mention heat equation, Schrödinger equation, certain population equations, Maxwell's equations, wave equation, and also seemingly unrelated problems such as delay equations, Markov processes or Boltzmann's equations.

In the present paper, we will continue the approach initiated by Perron for semilinear nonautonomous evolution equations of the form

$$\frac{dx}{dt} = A(t)x + f(t,x) \ . \tag{4}$$

Here $A(t)$ is a (possibly unbounded) linear operator acting on a real or complex Banach space $\mathbb{X}$ and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a (possibly nonlinear) continuous function. Following [1], we assume that the linear equation (3) is well-posed (i.e. there exists a continuous linear evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ such that for every $s \in \mathbb{R}_+$ and $x \in D(A(s))$, the function $x(t) = U(t,s)x$ is the uniquely determined solution of equation (3) satisfying $x(s) = x$). Then we can consider the **mild solution** of the semilinear equation (4) (defined on some interval $[s, s+\delta), \delta > 0$) as being the solution of the integral equation

$$x(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau, x(\tau))d\tau \quad , \quad t \geq s \ , \tag{5}$$

Furthermore, if we assume also that the nonlinear function $f(t,x)$ is jointly continuous with respect to t and x and Lipschitz continuous with respect to x (uniformly in $t \in \mathbb{R}_+$, and $f(t,0) = 0$ for all $t \in \mathbb{R}_+$) we can generate a (nonlinear) evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$, in the sense that the map $t \mapsto X(t,s)x : [s,\infty) \to \mathbb{X}$ is the unique solution of equation (6), for every $x \in \mathbb{X}$ and $s \in \mathbb{R}_+$.

Considering the *Green's operator* $(\mathbb{G}f)(t) = \int_0^t X(t,s)f(s)ds$ we prove that if the following conditions hold

- the map $\mathbb{G}f$ lies in $L^q(\mathbb{R}_+, \mathbb{X})$ for all $f \in L^p(\mathbb{R}_+, \mathbb{X})$, and
- $\mathbb{G} : L^p(\mathbb{R}_+, \mathbb{X}) \to L^q(\mathbb{R}_+, \mathbb{X})$ is Lipschitz continuous, i.e. there exists $K > 0$ such that

$$\|\mathbb{G}f - \mathbb{G}g\|_q \leq K\|f - g\|_p \ , \text{ for all } f, g \in L^p(\mathbb{R}_+, \mathbb{X}) \ ,$$

then the above mild solution will have an exponential decay.

It is worth to note that, although the autonomous case (i.e. time invariant evolution equations), was much more analyzed than the nonautonomous case, the latter one often arises quite naturally, not only in physics and mechanics, but also in the mathematical theory of differential equations when one linearizes an autonomous equation along a nonstationary solution. For particular cases of autonomous evolution equations arising from the linearization along a compact invariant manifold it has been shown (see e.g. [43]) that one can define a skew-product semiflow which allows to apply the methods of classical dynamical systems to the underlying time-dependent equations.

For the case of a time-invariant linear part of equation (4) the existence problem for solutions has been investigated by many authors (see e.g. [18, 19, 24, 33, 34, 35, 36, 48] and the references therein).

2. Semilinear evolution equations. Examples

First, let us recall some notations and definitions. Throughout this paper, $\mathbb{X}$ will denote a Banach space, $\mathbb{R}$ the set of all real numbers, $\mathbb{R}_+$ the subset of all nonnegative real numbers and put $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$. If $\mathbb{Y}$ denotes also a Banach space, then the set of all maps $T : \mathbb{X} \to \mathbb{Y}$ such that

$$\|T\|_{lip} := \inf\{M > 0 : \|Tx - Ty\| \leq M\|x - y\| \ , \text{ for all } x, y \in \mathbb{X}\} \ < \ \infty \ .$$

will be denoted by $Lip(\mathbb{X}, \mathbb{Y})$. Also, if $\mathbb{X} = \mathbb{Y}$ we will put simply $Lip(\mathbb{X})$ instead of $Lip(\mathbb{X}, \mathbb{X})$. It is easy to see that $(Lip(\mathbb{X}), \|\cdot\|_{lip})$ is a seminormed vector space which has the property

$$\|T \circ S\|_{lip} \leq \|T\|_{lip}\|S\|_{lip} \text{ for all } T, S \in Lip(\mathbb{X}) \ .$$

For a given interval $\mathbb{J}$ of the real line, we denote by

$$L^p(\mathbb{J}, \mathbb{X}) = \{f : \mathbb{J} \to \mathbb{X} : f \text{ is measurable and } \int_{\mathbb{J}} \|f(t)\|^p dt < \infty\} \quad ,$$

for all $p \in [1, \infty)$ and by

$$L^\infty(\mathbb{J}, \mathbb{X}) = \{f : \mathbb{J} \to \mathbb{X} \ : \ f \text{ is measurable and } \operatorname*{ess\,sup}_{t\in\mathbb{J}} \|f(t)\| < \infty\}.$$

It is well-known that $L^p(\mathbb{J},\mathbb{X}), L^\infty(\mathbb{J},\mathbb{X})$ are Banach spaces endowed with the norms

$$\|f\|_p = \left(\int_{\mathbb{J}} \|f(t)\|^p dt\right)^{1/p} \quad ,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{J}} \|f(t)\| \quad ,$$

respectively.

Definition 2.1. A family $\{X(t,s)\}_{(t,s)\in\Delta}$ of (possibly nonlinear) operators acting on $\mathbb{X}$ is called an **evolution family** if it satisfies the following conditions:

- (e_1) $X(t,t)x = x$ for all $t \geq 0$ and $x \in \mathbb{X}$;
- (e_2) $X(t,s) = X(t,r)\circ X(r,s)$ for all $t \geq r \geq s \geq 0$.

Such an evolution family is called **continuous** if there exist $M, \omega > 0$ such that

- (e_3) $\|X(t,s)\|_{lip} \leq Me^{\omega(t-s)}$
- (e_4) $X(t,s)x$ is jointly continuous with respect to t, s and x..

If the operators $X(t,s)$ are linear, then by (e_3) they are also bounded and the family will be called a **linear evolution family**.

Remark 2.2. Condition (e_3) is equivalent with the existence of some locally bounded function φ such that

(e_3') $\|X(t,s)x - X(t,s)y\| \leq \varphi(t-s)\|x-y\|$ for all $x, y \in \mathbb{X}$.

Indeed, if (e_3) holds one can take $\varphi(t) = Me^{\omega t}$. Conversely, the constants $M = \sup_{t\in[0,1]}\varphi(t)$ and $\omega = \max\{1, \ln\varphi(1)\}$ satisfy (e_3).

Definition 2.3. The linear equation (3) is said to be **well-posed** if there exists a continuous linear evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ such that for every $s \in \mathbb{R}_+$ and $x \in D(A(s))$, the function $x(t) = U(t,s)x$ is the uniquely determined solution of equation (3) satisfying $x(s) = x$.

Definition 2.4. Suppose the linear equation (3) is well-posed. Then, every solution $x(t)$ (defined on some interval $[s, s+\delta), \delta > 0$) of the integral equation

$$x(t) = U(t,s)x + \int_s^t U(t,\tau)f(\tau, x(\tau))d\tau \quad , \quad t \geq s \ , \tag{6}$$

is called a **mild solution** of the semilinear equation (4) starting from x at $t = s$. Furthermore, **equation** (4) **is said to generate an evolution family** $\{X(t,s)\}_{(t,s)\in\Delta}$ if for every $x \in \mathbb{X}$ and $s \in \mathbb{R}_+$, the map $t \mapsto X(t,s)x : [s,\infty) \to \mathbb{X}$ is the unique solution of equation (6).

Proposition 2.5. *Suppose the following conditions are satisfied:*

- (i) *The linear equation* (3) *is well-posed;*
- (ii) *The nonlinear function $f(t,x)$ is jointly continuous with respect to t and x and Lipschitz continuous with respect to x, uniformly in $t \in \mathbb{R}_+$, and $f(t,0) = 0$ for all $t \in \mathbb{R}_+$.*

Then, the semilinear equation (4) *generates a continuous evolution family.*

Proof. Using standard arguments, see for instance [44] it can be shown that the equation (4) generates an evolution family. Moreover, from [44], it follows that $X(t,s)x$ is jointly continuous with respect to t, s and x. We show briefly bellow that $X(t,s)x$ also fulfill condition (e_3) in Definition 2.1. We have that

$$\|X(t,s)x - X(t,s)y\| \leq \|U(t,s)x - U(t,s)y\| + \int_s^t \|U(t,\xi)\|\,\|f(\xi, X(\xi,s)x) - f(\xi, X(\xi,s)y)\|d\xi$$
$$\leq Ke^{\omega(t-s)}\|x-y\| + \int_s^t Ke^{\omega(t-\xi)}L\|X(\xi,s)x - X(\xi,s)y\|d\xi \ ,$$

where L is a Lipschitz constant of $f(t,x)$ with respect to x and K, ω stem from the well-posedness of the linear equation (3) and for convenience choose ω to be positive. Applying Gronwall's Lemma we get

$$\|X(t,s)x - X(t,s)y\| \leq Ke^{(\omega+KL)(t-s)}\|x-y\| \ ,$$

for any $(t,s) \in \Delta$ and $x, y \in X$.

$\square$

Remark 2.6. It is worth to note that one of the goals with respect to the asymptotic behavior of solutions of the equation (4) is also to point out conditions for that equation to admit invariant (stable, unstable or center) manifolds (see, e.g., [1, 3, 11, 14, 16, 17, 29, 45]). As far as we know, the most popular conditions for the existence of invariant manifolds are the exponential stability, dichotomy of the linear part (3) and the uniform Lipschitz continuity of the nonlinear part $f(t,x)$ with sufficiently small Lipschitz constants (i.e., $\|f(t,x)-f(t,y)\| \leq M\|x-y\|$ for M small enough). Moreover, the manifolds considered in the existing literature are mostly constituted by trajectories of solutions bounded on the positive (or negative) half-line. We refer the reader to [1, 3, 14, 16, 17, 29, 45] and references therein for more details on this topic.

Example 2.7. For a given continuous map $h : \mathbb{R} \to [1/2, 1]$ consider the differential equation on $\mathbb{R}$

$$\dot{u}(t) = h(u(t)) \ . \tag{7}$$

We claim that this equation leads to a continuous evolution family.

Indeed, consider the map $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(t) = \int_0^t \frac{ds}{h(s)} \quad .$$

One can easily check that

$$|u-v| \leq |H(u)-H(v)| \leq 2|u-v|$$

for all $u, v \in \mathbb{R}$. It follows that H is bijective and so it is easy to see that

$$X(t,s) : \mathbb{R} \to \mathbb{R} \quad , \quad X(t,s)x = H^{-1}(t-s+H(x)) \tag{8}$$

is an evolution family which has the property (e_4). Also, we have

$$\begin{aligned} |X(t,s)x - X(t,s)y| =& |H^{-1}(t-s+H(x)) - H^{-1}(t-s+H(y))| \\ \leq& |(t-s+H(x)) - (t-s+H(y))| \\ \leq& |x-y| \quad , \end{aligned}$$

for all $(t,s) \in \Delta$ and all $x \in \mathbb{R}$. Thus we obtain that $\{X(t,s)\}_{(t,s)\in\Delta}$ is a continuous evolution family on $\mathbb{R}$.

Example 2.8. Consider $h : \mathbb{R} \to \mathbb{R}$ continuously differentiable with $h' \in L^\infty(\mathbb{R}, \mathbb{R})$ and the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + h(u(x,t)) \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 \end{cases} \tag{9}$$

If we denote $x(t) = u(\,\cdot\,, t)$, the problem (9) is equivalent to

$$\dot{x}(t) = Ax(t) + f(x(t)), \tag{10}$$

where $A : D(A) \subset L^2([0,1], \mathbb{R}) \to L^2([0,1], \mathbb{R})$, $D(A)$ is defined as the set of all functions $z \in L^2([0,1], \mathbb{R})$ such that z, z' are absolutely continuous with $z'' \in L^2([0,1], \mathbb{R})$ and $z'(0) = z'(1) = 1$, and for each $z \in D(A)$, we define $Az = z''$.

Also we consider $B : L^2([0,1], \mathbb{R}) \to L^2([0,1], \mathbb{R})$ which is given by $Bz = h \circ z$. It is well known that A generates a strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$ in $L^2([0,1], \mathbb{R})$. And it is easy to check that B is Lipschitz continuous.

By Example 2.7 it is clear that the equation (10), as an abstract variant of (9), generates a continuous evolution family.

For $\{X(t,s)\}_{(t,s)\in\Delta}$ an evolution family, the *trajectory* determined by $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{X}$ will be denoted by

$$u_{t_0,x} : \mathbb{R}_+ \to \mathbb{X} \quad , \quad u_{t_0,x}(t) = \begin{cases} X(t,t_0)x & , \ t \geq t_0 \\ 0 & , \ 0 \leq t \leq t_0 \end{cases} \quad . \tag{11}$$

Definition 2.9. An evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$ is said to be

(u.e.s.) **uniformly exponentially stable**, if there exist $N, \nu > 0$ such that

$$\|X(t,s)\|_{lip} \leq N e^{-\nu(t-s)} \ , \ \text{for all } (t,s) \in \Delta \ ; \tag{12}$$

(u.s.) **uniformly stable** if there exists $N > 0$ such that

$$\|X(t,s)\|_{lip} \leq N \ , \ \text{for all } (t,s) \in \Delta \ ; \tag{13}$$

(a.s.) **asymptotically stable** if all its trajectories are decaying to zero, i.e.

$$\lim_{t\to\infty} u_{t_0,x_0}(t) = 0 \ , \ \text{for all } t_0 \in \mathbb{R}_+ \text{ and } x_0 \in \mathbb{X} \ . \tag{14}$$

We will need in the next the following additional lemma.

Lemma 2.10. *Let* $\{X(t,s)\}_{(t,s)\in\Delta}$ *be a continuous evolution family on* $\mathbb{X}$. *Then, the function* $s \mapsto X(t,s)f(s) : \mathbb{R}_+ \to \mathbb{X}$ *is locally integrable provided that* $f : \mathbb{R}_+ \to \mathbb{X}$ *is locally integrable.*

Proof. This claim follows easily using conditions (e_2), (e_3) and (e_4) of $\{X(t,s)\}_{(t,s)\in\Delta}$:

$$\begin{aligned} \int_0^t \|X(t,s)f(s) - X(t,0)f(0)\| ds &\leq \int_0^t \|X(t,s)f(s) - X(t,s)X(s,0)f(0)\| ds \\ &\leq M e^{\omega t} \int_0^t \left(\|f(s)\| + \|X(s,0)f(0)\|\right) ds \end{aligned} \tag{15}$$

which holds for any $t \in \mathbb{R}_+$. □

Given a continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$, we will denote by $\mathbb{G}$ the *Green's operator*

$$\mathbb{G} : L^1_{loc}(\mathbb{R}_+, \mathbb{X}) \to L^1_{loc}(\mathbb{R}_+, \mathbb{X}) \quad , \quad (\mathbb{G}f)(t) = \int_0^t X(t,s)f(s) ds \quad . \tag{16}$$

(where $L^1_{loc}(\mathbb{R}_+, \mathbb{X})$ denotes the space of all locally integrable functions from $\mathbb{R}_+$ into $\mathbb{X}$). Set now

$$\mathcal{A}_X = \{\chi_{[a,b]} u_{t_0,x} : x \in \mathbb{X}, \ t_0 \geq 0, \ 0 \leq a \leq b\} \ ,$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. Then, A_X is contained in $L^p(\mathbb{R}_+, \mathbb{X})$, for every $p \in [1, \infty]$.

Definition 2.11. The pair $(L^p(\mathbb{R}_+, \mathbb{X}), L^q(\mathbb{R}_+, \mathbb{X}))$ is said to be **admissible to** $\{X(t,s)\}_{(t,s)\in\Delta}$ if the following conditions hold

- the map $\mathbb{G}f$ lies in $L^q(\mathbb{R}_+, \mathbb{X})$ for all $f \in L^p(\mathbb{R}_+, \mathbb{X})$, and
- $\mathbb{G} : L^p(\mathbb{R}_+, \mathbb{X}) \to L^q(\mathbb{R}_+, \mathbb{X})$ is Lipschitz continuous, i.e. there exists $K > 0$ such that

$$\|\mathbb{G}f - \mathbb{G}g\|_q \leq K \|f - g\|_p \ , \ \text{for all } f, g \in L^p(\mathbb{R}_+, \mathbb{X}) \ .$$

3. Main results

Lemma 3.1. *Let* $h \in L^q(\mathbb{R}_+, \mathbb{R})$, $q \in [1, \infty]$ *such that* $h(0) \geq 0$. *If*

$$h(r) \leq m\, h(t) \quad , \ \textit{for all } r \in [t, t+1] \textit{ and for all } t \geq 0 \ ,$$

then, $h \in L^\infty(\mathbb{R}_+, \mathbb{R})$ *and* $\|h\|_\infty \leq m h(0) + m\|h\|_q$.

Proof. Let $r \geq 1$. Since $h(r) \leq m h(t)$ for all $t \in [r-1, r]$, it follows that

$$h(r) \ \leq \ m \int_{r-1}^r h(t)\, dt \ \leq \ m \|h\|_q \ .$$

If $r \in [0,1]$, from the hypothesis we have that $h(r) \leq m\, h(0)$. Therefore,

$$h(r) \leq m(h(0) + \|h\|_q) \ ,$$

for every $r \geq 0$, and the above claim follows immediately. □

Lemma 3.2. *Let* $g : \Delta \to \mathbb{R}_+$ *be a function with the following properties:*

i) $g(t, t_0) \leq g(t,s) g(s, t_0)$, *for all* $t \geq s \geq t_0$;

ii) *there exist* $M, d > 0$ *and* $c \in (0,1)$ *satisfying*

$$g(t,t_0) \leq M, \text{ for all } t \in [t_0, t_0+d]\,,\ t_0 \geq 0 \text{ and}$$
$$g(t_0+d, t_0) \leq c\,,\ \text{for all } t_0 \geq 0\ .$$

Then, there exist $N, \nu > 0$ *such that*

$$g(t,t_0) \leq Ne^{-\nu(t-t_0)}\ ,\ \text{for all}\, t \geq t_0 \geq 0\ .$$

Proof. Let $(t,t_0) \in \Delta$ and $n = \left[\frac{t-t_0}{d}\right]$. Then, we have that

$$\begin{aligned} g(t,t_0) &\leq g(t, t_0+nd)g(t_0+nd, t_0) \leq g(t,t_0+nd)c^n \\ &\leq Mc^n = Me^{-\nu nd} \leq Ne^{-\nu(t-t_0)}\ , \end{aligned}$$

where $\nu = -\frac{1}{d}\ln c$, $N = Me^{\nu d}$. □

Next we define the functions $a_p, b_p : \mathbb{R}_+ \to \mathbb{R}$ given by

$$a_p(t) = \begin{cases} t^{1-\frac{1}{p}} & ,\quad p \in [1,\infty) \\ t & ,\quad p = \infty \end{cases}\ , b_p(t) = \|\chi_{[0,t]}\|_p = \begin{cases} t^{\frac{1}{p}} & ,\quad p \in [1,\infty) \\ 1 & ,\quad p = \infty. \end{cases}$$

Remark 3.3. One can easily check that

$$\int_{t_0}^{t_0+t} \|f(s)\|ds \leq a_p(t)\|f\|_p\ ,$$

for all $t, t_0 \geq 0$ and $f \in L^p(\mathbb{R}_+, \mathbb{X})$.

Theorem 3.4. *Let* $\{X(t,s)\}_{(t,s)\in\Delta}$ *be a continuous evolution family and* $p, q \in [1,\infty]$ *such that* $(p,q) \neq (1,\infty)$. *If the pair* $(L^p(\mathbb{R}_+,\mathbb{X}), L^q(\mathbb{R}_+,\mathbb{X}))$ *is admissible to* $\{X(t,s)\}_{(t,s)\in\Delta}$ *, then* $\{X(t,s)\}_{(t,s)\in\Delta}$ *is uniformly exponentially stable.*

Proof. Let $t_0 \geq 0,\ x_1,\ x_2 \in \mathbb{X}$ and $f_1,\ f_2 : \mathbb{R}_+ \to \mathbb{X}$ given by

$$f_i(t) = \begin{cases} X(t,t_0)x_i & ,\ t \in [t_0, t_0+1] \\ 0 & ,\ t \in \mathbb{R}_+ \setminus [t_0, t_0+1] \end{cases}\quad ,\ i = 1,2\ . \tag{17}$$

Clearly, $f_1, f_2 \in \mathcal{A}_X$ with $\|f_1 - f_2\|_p \leq Me^{\omega}\|x_1 - x_2\|$ and

$$(\mathbb{G}f_i)(t) = \int_0^t X(t,s)f_i(s)ds = \int_{t_0}^{t_0+1} X(t,s)X(s,t_0)x_i ds = X(t,t_0)x_i = u_{t_0,x_i}(t)\ ,$$

for all $t \geq t_0+1$, $i = 1,2$. For $t \in [t_0, t_0+1]$, we have that

$$\|u_{t_0,x_1}(t) - u_{t_0,x_2}(t)\| \leq \|X(t,t_0)\|_{lip}\|x_1 - x_2\| \leq Me^{\omega}\|x_1 - x_2\|\ .$$

It follows that $u_{t_0,x_1} - u_{t_0,x_2} \in L^q(\mathbb{R}_+, \mathbb{X})$ and

$$\begin{aligned} \|u_{t_0,x_1} - u_{t_0,x_2}\|_q &\leq Me^{\omega}\|x_1 - x_2\| + \|(\mathbb{G}f_1 - \mathbb{G}f_2)\,\chi_{[t_0+1,\infty)}\|_q \\ &\leq Me^{\omega}\|x_1 - x_2\| + \|\mathbb{G}f_1 - \mathbb{G}f_2\|_q \\ &\leq Me^{\omega}\|x_1 - x_2\| + K\|f_1 - f_2\|_p \\ &\leq (K+1)Me^{w}\|x_1 - x_2\|. \end{aligned} \tag{18}$$

Let us define the map $h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(t) = \|u_{t_0,x_1}(t_0+t) - u_{t_0,x_2}(t_0+t)\|$. Then, $h \in L^q(\mathbb{R}_+, \mathbb{R})$ with $\|h\|_q = \|u_{t_0,x_1} - u_{t_0,x_2}\|_q \leq (K+1)Me^{w}\|x_1 - x_2\|$, and

$$\begin{aligned} h(r) &= \|X(t_0+r, t_0+t)X(t_0+t, t_0)x_1 - X(t_0+r, t_0+t)X(t_0+t,t_0)x_2\| \\ &\leq \|X(t_0+r, t_0+t)\|_{lip}\|X(t_0+t,t_0)x_1 - X(t_0+t,t_0)x_2\| \\ &\leq Me^{\omega(r-t)}h(t) \leq Me^{\omega}h(t)\quad ,\quad 0 \leq t \leq r \leq t+1\ . \end{aligned}$$

By Lemma 3.1 we obtain that $h \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and

$$\|h\|_\infty \leq Me^{\omega}\|h\|_q + Me^{\omega}h(0) \leq (K+1)M^2e^{2\omega}\|x_1 - x_2\| + Me^{\omega}\|x_1 - x_2\|$$

Now we have that there exists $C > 0$ such that

$$\|X(t,s)x_1 - X(t,s)x_2\| \leq C\|x_1 - x_2\| \ ,$$

for all $(t,s) \in \Delta$ and $x_1 x_2 \in \mathbb{X}$ and hence

$$\|X(t,s)\|_{lip} \leq C \ , \quad \text{for all} \quad (t,s) \in \Delta \ . \tag{19}$$

Consider again $x_1, x_2 \in \mathbb{X}$, $t_0 \geq 0$, $\delta > 0$, $f_3, f_4 : \mathbb{R}_+ \to \mathbb{X}$ given by

$$f_i(t) = \begin{cases} X(t,t_0)x_{i-2} & , \ t \in [t_0, t_0+\delta] \\ 0 & , \ t \in \mathbb{R}_+ \setminus [t_0, t_0+\delta] \end{cases} \quad , \ i = 3,4 \ . \tag{20}$$

Then $f_3, f_4 \in \mathcal{A}_X$ with $\|f_3 - f_4\|_p \leq Cb_p(\delta)\|x_1 - x_2\|$ and

$$(\mathbb{G}f_i)(t) = \int_0^t X(t,s)f_i(s)ds = \int_{t_0}^t X(t,s)X(s,t_0)x_{i-2}\, ds = (t-t_0)U(t,t_0)x_{i-2}$$

for all $t \in [t_0, t_0+\delta]$, $i = 3,4$. Using the last equality we have that

$$\begin{aligned} \frac{\delta^2}{2}\|X(t_0+\delta,t_0)x_1 - X(t_0+\delta,t_0)x_2\| &= \int_{t_0}^{t_0+\delta} (t-t_0)\|X(t_0+\delta,t_0)x_1 - X(t_0+\delta,t_0)x_2\|dt \\ &\leq C \int_{t_0}^{t_0+\delta} (t-t_0)\|X(t,t_0)x_1 - X(t,t_0)x_2\|dt \\ &= C \int_{t_0}^{t_0+\delta} \|(\mathbb{G}f_3)(t) - (\mathbb{G}f_4)(t)\|dt \\ &\leq Ca_q(\delta)\|\mathbb{G}f_3 - \mathbb{G}f_4\|_q \\ &\leq KCa_q(\delta)\|f_3 - f_4\|_p \\ &\leq KC^2 a_q(\delta)b_p(\delta)\|x_1 - x_2\| \\ &= \frac{KC^2\delta^2}{a_p(\delta)b_q(\delta)}\|x_1 - x_2\| \ . \end{aligned} \tag{21}$$

It follows that

$$\|X(t_0+\delta,t_0)\|_{lip} \leq \frac{2KC^2}{a_p(\delta)b_q(\delta)} \ ,$$

for all $t_0 \geq 0$, $\delta > 0$. Since $(p,q) \neq (1,\infty)$, we have that $\lim\limits_{\delta\to\infty} a_p(\delta)b_q(\delta) = \infty$, and therefore we can choose $d > 0$ such that

$$\|X(t_0+d,t_0)\|_{lip} \leq \frac{1}{2} \ , \tag{22}$$

for all $t_0 \geq 0$. Applying Lemma 3.2 to the map $g : \Delta \to \mathbb{R}_+$, defined by $g(t,s) = \|U(t,s)\|_{lip}$ we obtain that $\{X(t,s)\}_{(t,s)\in\Delta}$ is uniformly exponentially stable.

□

Theorem 3.5. *The pair $(L^1(\mathbb{R}_+,\mathbb{X}), L^\infty(\mathbb{R}_+,\mathbb{X}))$ is admissible to the continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$ if and only if the following statements hold*

- (i) *there exists $\psi \in L^1(\mathbb{R}_+,\mathbb{X})$ such that $\mathbb{G}\psi \in L^\infty(\mathbb{R}_+,\mathbb{X})$, and*
- (ii) *there exists $N > 0$ such that $\|X(t,s)\|_{lip} \leq N$, for all $(t,s) \in \Delta$.*

Proof. The *necessity* follows from the proof of Theorem 3.4 since we proved (19) without the $(p,q) \neq (1,\infty)$ assumption.

Sufficiency. Let $f \in L^\infty(\mathbb{R}_+, \mathbb{X})$. Since $\psi \in L^1(\mathbb{R}_+, \mathbb{X})$ and $f \in L^1(\mathbb{R}_+, \mathbb{X})$, we have that

$$\begin{aligned} \|(\mathbb{G}\psi)(t) - (\mathbb{G}f)(t)\| &\leq \int_0^t \|X(t,s)\phi(s) - X(t,s)f(s)\|ds \\ &\leq \int_0^t \|X(t,s)\|_{lip}\|\psi(s) - f(s)\|ds \\ &\leq N\int_0^t \|\psi(s) - f(s)\|ds \\ &\leq N\|\psi - f\|_1 \quad , \end{aligned} \tag{23}$$

for all $t \geq 0$, which implies that $\mathbb{G}f \in L^\infty(\mathbb{R}_+, \mathbb{X})$.

It remains to prove that $\mathbb{G} : L^1(\mathbb{R}_+, \mathbb{X}) \to L^\infty(\mathbb{R}_+, \mathbb{X})$ is Lipschitz continuous. But, using similar arguments as in (23), we obtain that

$$\|\mathbb{G}f_1 - \mathbb{G}g\|_{L^\infty(\mathbb{R}_+,\mathbb{X})} \leq N\|f - g\|_{L^1(\mathbb{R}_+,\mathbb{X})} \ , \tag{24}$$

for all $f, g \in L^1(\mathbb{R}_+, \mathbb{X})$. □

Remark 3.6. Theorem 3.5 extends a similar result for linear equations (see e.g. [7, 25]). Note that in the linear case, condition (ii) is automatically satisfied.

In the following theorem we try to answer concerns regarding the converse of what was obtained in Theorem 3.4. With elementary arguments we can show that the admissibility of the pair $(L^p(\mathbb{R}_+, \mathbb{X}), L^q(\mathbb{R}_+, \mathbb{X}))$ with $p \leq q$ is a necessary condition for the uniform exponential stability of a continuous evolution family. The idea is based on the use of Fubini's theorem, Hölder's inequality and the observation that if $f \in L^p(\mathbb{R}_+, \mathbb{X}) \cap L^q(\mathbb{R}_+, \mathbb{X})$, then $f \in L^r(\mathbb{R}_+, \mathbb{X})$ with $\|f\|_r \leq \max\{\|f\|_p, \|f\|_q\}$, for any $1 \leq p \leq r \leq q \leq \infty$.

Theorem 3.7. *Let $\{X(t,s)\}_{(t,s)\in\Delta}$ be a continuous evolution family and $1 \leq p \leq q \leq \infty$.*
Then, the pair $(L^p(\mathbb{R}_+, \mathbb{X}), L^q(\mathbb{R}_+, \mathbb{X}))$ is admissible to $\{X(t,s)\}_{(t,s)\in\Delta}$ provided that

i) *there is a function $\psi \in L^p(\mathbb{R}_+, \mathbb{X})$ such that $\mathbb{G}\psi \in L^q(\mathbb{R}_+, \mathbb{X})$, and*
ii) *$\{X(t,s)\}_{(t,s)\in\Delta}$ is uniformly exponentially stable.*

Proof. Let $N, \nu > 0$ be like in Definition 2.9. Fix arbitrarily $f, g \in L^p(\mathbb{R}_+, \mathbb{X})$. We have that

$$\begin{aligned} \|(\mathbb{G}f)(t) - (\mathbb{G}g)(t)\| &\leq \int_0^t \|X(t,s)(f(s) - g(s))\|ds \leq \int_0^t \|X(t,s)\|_{lip}\|f(s) - g(s)\|ds \\ &\leq N\int_0^t e^{-\nu(t-s)}\|f(s) - g(s)\|ds \end{aligned}$$

Consider $h, H : \mathbb{R}_+ \to \mathbb{R}_+$ given by $h(t) = \|f(t) - g(t)\|$ and

$$H(t) = \int_0^t e^{-\nu(t-s)}h(s)ds \tag{25}$$

for any $t \in \mathbb{R}_+$. In what follows, we will prove that if $h \in L^p(\mathbb{R}_+, \mathbb{R})$ then $H \in L^q(\mathbb{R}_+, \mathbb{R})$ (in the hypothesis $1 \leq p \leq q \leq \infty$).

Case 1. If $p = \infty$, then $q = \infty$ and since

$$H(t) \leq \int_0^t e^{-\nu(t-s)}\|h\|_\infty ds \leq \|h\|_\infty \int_0^t e^{-\nu\tau}d\tau \leq \frac{1}{\nu}\|h\|_\infty \ ,$$

for all $t \in \mathbb{R}_+$, it follows that $H \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\|H\|_\infty \leq \frac{1}{\nu}\|h\|_\infty$.

Case 2. If $p = 1$, note that

$$H(t) = \int_0^t e^{-\nu(t-s)}h(s)ds \leq \int_0^t h(s)ds \leq \|h\|_1 \ ,$$

for all $t \in \mathbb{R}^+$ and thus $H \in L^\infty(\mathbb{R}_+, \mathbb{R})$ with $\|H\|_\infty \leq \|h\|_1$. Also, using Fubini's theorem we have that

$$\begin{aligned}\int_0^\infty H(t)dt &= \int_0^\infty \int_0^t e^{-\nu(t-s)} h(s)\, ds\, dt = \int_0^\infty \int_s^\infty e^{-\nu(t-s)} h(s)\, dt\, ds \\ &= \int_0^\infty e^{\nu s} h(s) \int_s^\infty e^{-\nu t} dt\, ds = \int_0^\infty e^{\nu s} h(s) \frac{e^{-\nu s}}{\nu} ds \\ &= \frac{1}{\nu}\|h\|_1 ,\end{aligned}$$

which implies that $H \in L^1(\mathbb{R}_+, \mathbb{R})$ with $\|H\|_1 \leq \frac{1}{\nu}\|h\|_1$. Then, $H \in L^q(\mathbb{R}_+, \mathbb{R})$ and $\|H\|_q \leq \max\{1, 1/\nu\}\|h\|_1$.

Case 3. If $p \in (1, \infty)$, let $p' \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and let $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$. For any $t \in \mathbb{R}_+$, we can write down

$$H(t) \leq \left(\int_0^t h(s)^p ds\right)^{1/p} \left(\int_0^t e^{-\nu p' \tau} d\tau\right)^{1/p'} \leq \frac{1}{(\nu p')^{1/p'}} \|h\|_p .$$

We obtained that $H \in L^\infty(\mathbb{R}_+, \mathbb{R})$ and $\|H\|_\infty \leq (\nu p')^{-1/p'}\|h\|_p$. Next, we prove that $H \in L^p(\mathbb{R}_+, \mathbb{R})$. Via Hölder's inequality, we have that

$$\begin{aligned}\int_0^t e^{-\nu(t-s)} h(s) ds &= \int_0^t e^{-\nu\alpha(t-s)} e^{-\nu\beta(t-s)} h(s) ds \\ &\leq \left(\int_0^t e^{-\nu\alpha p'(t-s)} ds\right)^{1/p'} \left(\int_0^t e^{-\nu\beta p(t-s)} h(s) ds\right)^{1/p} \\ &\leq \left[\left(\frac{1}{\nu\alpha p'}\right)^{p-1} \int_0^t e^{-\nu\beta p(t-s)} h(s) ds\right]^{1/p} .\end{aligned}$$

Then, denoting $C := (\nu\alpha p')^{1-p} > 0$, we can write down

$$\begin{aligned}\int_0^\infty H(t)^p dt &= \int_0^\infty \left(\int_0^t e^{-\nu(t-s)} h(s) ds\right)^p dt \leq C \int_0^\infty \int_0^t e^{-\nu\beta p(t-s)} h(s)^p ds\, dt \\ &\leq \frac{C}{\nu\beta p}\|h\|_p^p\end{aligned}$$

(the last step follows similarly to *Case 2*, using Fubini's theorem). From here, it follows that $H \in L^p(\mathbb{R}_+, \mathbb{R})$ with $\|H\|_p \leq C^{1/p}(\nu\beta p)^{-1/p}\|h\|_p$. Therefore, $H \in L^q(\mathbb{R}_+, \mathbb{R})$ and we have that $\|H\|_q \leq \max\{(\nu p')^{-1/p'}, C^{1/p}(\nu\beta p)^{-1/p}\}\|h\|_p$.

In any case, we obtained that $H \in L^q(\mathbb{R}_+, \mathbb{R})$ (provided that $h \in L^p(\mathbb{R}_+, \mathbb{R})$) and the existence of some $K > 0$ (independent of h) such that $\|H\|_q \leq K\|h\|_p$.

To complete the proof, note that from (a) we have that $\psi - f \in L^p(\mathbb{R}_+, \mathbb{X})$ and $\mathbb{G}\psi \in L^q(\mathbb{R}_+, \mathbb{X})$; in virtue of all above we obtain that $\mathbb{G}\psi - \mathbb{G}f \in L^q(\mathbb{R}_+, \mathbb{X})$ (no matter how we take $f \in L^p(\mathbb{R}_+, \mathbb{X})$). Moreover, we can state that $\|\mathbb{G}f - \mathbb{G}g\|_q \leq K\|f - g\|_p$, for all $f, g \in L^p(\mathbb{R}_+, \mathbb{X})$. Hence, $\mathbb{G} \in Lip(L^p(\mathbb{R}_+, \mathbb{X}), L^q(\mathbb{R}_+, \mathbb{X}))$. □

Theorem 3.8. *If the pair $(L^1(\mathbb{R}_+, \mathbb{X}), L_0^\infty(\mathbb{R}_+, \mathbb{X}))$ is admissible to the continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$, then $\{X(t,s)\}_{(t,s)\in\Delta}$ is asymptotically stable.*

Proof. Let $x \in \mathbb{X}$, $t_0 \in \mathbb{R}_+$ and consider the function $f : \mathbb{R}_+ \to \mathbb{X}$

$$f(t) = \begin{cases} X(t, t_0)x & , \ t \in [t_0, t_0 + 1] \\ 0 & , \ t \in \mathbb{R}_+ \setminus [t_0, t_0 + 1] \end{cases} .$$

We have that $f \in L^1(\mathbb{R}_+, \mathbb{X})$ and note that

$$(\mathbb{G}f)(t) = \int_{t_0}^{t_0+1} X(t, s)X(s, t_0)x ds = u_{t_0,x}(t) ,$$

for all $t \geq t_0 + 1$. Since $\mathbb{G}f \in L_0^\infty(\mathbb{R}_+, \mathbb{X})$, we get that $\lim_{t\to\infty} u_{t_0,x}(t) = 0$.

□

4. Applications

In this section as a model for applications of the obtained results in the previous section we consider equations of the form

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} + g(t,u(t,x)), \quad t>0, x\in(0,\pi),\\ \frac{\partial u(t,x)}{\partial t} &= 0, \quad x=0,\pi,\end{aligned}$$

where $u(t,x)$ is a scalar function of $(t,x)\in\mathbb{R}^+\times\mathbb{R}$, $g(t,y)$ is uniformly Lipschitz continuous in $y\in\mathbb{R}$, and $g(\cdot,0)\in L^r(\mathbb{R}_+,\mathbb{X})$ with $1\le r<\infty$. This equation can be re-written in the following abstract form in a Banach space $\mathbb{X}$

$$\frac{d}{dt}u(t) = \Delta u(t) + G(t,u(t)), \tag{26}$$

where $X=\{v\in W^{2,2}(0,\pi): v'=0 \text{ at } x=0,\pi\}$, $\Delta v = \partial^2/\partial x^2$ on X, $G(t,u(t)) = g(t,u(t,\cdot))$. As is well known, (actually, an extension of) Δ generates a strongly continuous analytic semigroup in $\mathbb{X}$ that we denote by $(T(t))_{t\ge0}$. By a standard argument (see e.g. [1]), we can prove that (26) generates a continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$ in $\mathbb{X}$ that is determined from the equation

$$X(t,s)x = T(t-s)x + \int_s^t T(t-\xi)G(\xi, X(\xi,s)x)d\xi, \quad \text{for all } t\ge s\ge 0. \tag{27}$$

Theorem 4.1. *Assume that the pair $(L^p(\mathbb{R}_+,\mathbb{X}), L^q(\mathbb{R}_+,\mathbb{X}))$ is admissible to the continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$. Then there exists a mild solution $u\in L^r(\mathbb{R}_+,\mathbb{X})$ of (26) that attracts all other mild solutions of the equations at exponential rate.*

Proof. Let us consider the evolution semigroup $(T^h)_{h\ge0}$ in $L^r(\mathbb{R}_+,\mathbb{X})$ associated with the linear equations $u'=\Delta u$, defined as

$$[T^h f](t) := \begin{cases} T(h)f(t-h), & \text{if } h\le t\\ 0, & \text{if } 0\le t<h,\end{cases} \tag{28}$$

for all $f\in L^r(\mathbb{R}_+,\mathbb{X})$. Since $1\le r<\infty$, this semigroup is strongly continuous (see e.g. [5, 26]). Let us denote by $\mathcal{L}$ the generator of this semigroup. As is well known (see e.g. [26]), $u\in D(\mathcal{L})$ and $\mathcal{L}u=-f$ if and only if

$$u(t) = \int_0^t T(t-\xi)f(\xi d\xi \quad t\ge 0. \tag{29}$$

Consider the operator $\mathcal{L}+\mathcal{G}$ on $L^r(\mathbb{R}_+,\mathbb{X})$, where $\mathcal{G}$ is the Nemytsky operator associated with G, defined as $L^r(\mathbb{R}_+,\mathbb{X}) :\ni \phi(\cdot)\mapsto G(\cdot,\phi(\cdot))\in L^r(\mathbb{R}_+,\mathbb{X})$. Note that under the assumption on g, the Nemytsky operator acts in $L^r(\mathbb{R}_+,\mathbb{X})$ as a Lipschitz continuous operator. So, in the same way as in [1], we can show that the operator $\mathcal{L}+\mathcal{G}$ associated with (26) generates a strongly continuous semigroup $(S(h))_{h\ge0}$ in $L^r(\mathbb{R}_+,\mathbb{X})$ that is referred to as the evolution semigroup associated with (26). Moreover, this semigroup is determined by

$$[S(h)f](t) := \begin{cases} X(t,t-h)f(t-h), & \text{if } h\le t\\ 0, & \text{if } 0\le t<h.\end{cases} \tag{30}$$

for all $f\in L^r(\mathbb{R}_+,\mathbb{X})$. By the admissibility of the pair $(L^p(\mathbb{R}_+,\mathbb{X}), L^q(\mathbb{R}_+,\mathbb{X}))$, $\{X(t,s)\}_{(t,s)\in\Delta}$ is exponentially stable. This implies the strict contraction of $S(h)$ for sufficiently large h. In fact,

$$\|S(h)\phi - S(h)\psi\|_r \quad \le \quad Ne^{-\alpha h}\|\phi-\psi\|_r$$

for all $\phi,\psi\in L^r(\mathbb{R}_+,\mathbb{X})$. Therefore, if h is sufficiently large $Ne^{-\alpha h}<1$. Let us fix a sufficiently large integer n_0. This yields that $S(n_0)$ has a unique fixed point $\varphi\in L^r$. Since $S(h)$ commutes with $S(n_0)$ for all $h\ge0$, it is easy to see that φ is the unique common fixed point for the entire semigroup $(S(h))_{h\ge0}$. This implies that $(\mathcal{L}+\mathcal{G})\varphi=0$. So, $\mathcal{L}\varphi = -\mathcal{G}\varphi$, and by the formula (29), we have

$$\varphi(t) = \int_0^t T(t-\xi)G(\xi,\varphi(\xi))d\xi, \quad t\ge 0.$$

This means that $\varphi \in L^r(\mathbb{R}_+, \mathbb{X})$ is a mild solution starting at zero of (26), so $\varphi(t) = X(t,0)0$. Now we show that this solution attracts all other solutions at exponential rate. In fact, every other solution starting, say, at $x \in \mathbb{X}$ is of the form $X(t,0)x$. Therefore,

$$\|X(t,0)x - X(t,0)0\| \le Ne^{-\alpha t}\|x\|, \quad t \ge 0.$$

This completes the proof of the theorem. □

Before concluding this section we give an application of Theorem 3.5

Proposition 4.2. *Let the pair $(L^1(\mathbb{R}_+, \mathbb{X}), L^\infty(\mathbb{R}_+, \mathbb{X}))$ be admissible to the continuous evolution family $\{X(t,s)\}_{(t,s)\in\Delta}$ (generated by (26)). Moreover, assume that $g(t,0) = 0$ for all $t \ge 0$. Then every mild solution u of (26) is bounded.*

Proof. Obviously, the $u(t) = 0$ is the trivial mild solution of (26). Therefore, for every mild solution $u(t) = X(t,s)x$, where $x \in \mathbb{X}$, we have

$$\|u(t)\| = \|X(t,s)x\| = \|X(t,s)x - X(t,s)0\| \le \|X(t,s)\|_{lip}\|x\| \le N\|x\| \quad \text{for all } t \ge 0,$$

where N is a positive number whose existence is guaranteed by Theorem 3.5. □

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Department of Mathematics and Philosophy, Columbus State University, Columbus, GA 31907, USA
E-mail address: `nguyen_minh2@columusstate.edu`

Department of Mathematics, Morgan State University, Baltimore, Maryland 21251, USA
E-mail address: `Gaston.N'Guerekata@morgan.edu`

Department of Mathematics, West University of Timisoara, Romania
E-mail address: `ciprian.preda@feaa.uvt.ro`